

STABILITY OF PERIODIC ORBITS OF MULTIDIMENSIONAL HAMILTONIAN SYSTEMS.

Application to systems with four degrees of freedom[♦]

CH. SKOKOS^{1,2}

¹ Research Center for Astronomy, Academy of Athens, 14 Anagnostopoulou str.,
GR-10673, Athens, Greece

² Department of Physics, University of Athens, Panepistimiopolis,
GR-15784, Zografos, Greece

ABSTRACT: We consider the problem of stability of periodic orbits of autonomous Hamiltonian systems with $N+1$ degrees of freedom or equivalently of $2N$ -dimensional symplectic maps, with $N \geq 1$. Following Skokos (2001) we classify the different stability types a periodic orbit can exhibit, introducing a new terminology which is perfectly suited for systems with many degrees of freedom, since it clearly reflects the configuration of the eigenvalues of the corresponding monodromy matrix, on the complex plane. The different stability types correspond to different regions of the N -dimensional parameter space \mathcal{S} , defined by the coefficients of the characteristic polynomial of the monodromy matrix. We apply our results to the well known cases of Hamiltonian systems with two and three degrees of freedom. We also describe in detail the different stability regions in the three-dimensional parameter space \mathcal{S} of a Hamiltonian system with four degrees of freedom or equivalently of a six-dimensional symplectic map.

1. Introduction

The stability type of periodic orbits of Hamiltonian systems influence significantly the dynamical behavior of the system, since non-periodic orbits near a stable periodic orbit are ordered, i.e. their evolution in time is similar to the behavior of the periodic orbit, while the unstable periodic orbits introduce chaotic behavior in the system.

The periodic orbits and their stability have been extensively studied in the last decades for autonomous Hamiltonian systems with 2 or 3 degrees of freedom, both theoretically and numerically (e.g. Hénon 1965, Broucke 1969, Hadjidemetriou 1975, Bountis 1981, Contopoulos & Magnenat 1985, Heggie 1985). Some work has been also done on systems with many degrees of freedom (Howard & MacKay 1987, Gautero & Froeschlé 1990, Howard & Dullin 1998), but a general theory in dimensions higher than three is still lacking. Most of the activity so far was focused on the derivation of stability boundaries for symplectic maps (Howard & MacKay 1987, Howard & Dullin 1998). Howard & MacKay (1987), based on the observation that the introduction of the stability indices reduces the characteristic polynomial which gives the eigenvalues of the monodromy matrix to a polynomial with half the original order, succeeded in obtaining results for the stability boundaries of symplectic maps of dimension as high as eight.

[♦] This work is dedicated to the memory of Dr. C. Polymilis who suggested the subject of the present paper; unfortunately he passed away before this work was completed.

In the present paper we mainly present some recent results on the stability of periodic orbits in multidimensional systems (Skokos 2001). We classify all the possible stability types in the general case of a Hamiltonian system with $N+1$ degrees of freedom, which corresponds to a $2N$ -dimensional symplectic map and introduce a new terminology for them. After systems of three degrees of freedom which have been studied extensively, the next step towards understanding instabilities in multidimensional systems is the case of four degrees of freedom. So we study in detail the case of a Hamiltonian system with four degrees of freedom which corresponds to a six-dimensional symplectic map. We note that the stability parameter space \mathcal{S} of such systems is three-dimensional and can be visualized.

The paper is organized as follows. In Section 2 we review briefly the basic theory of stability of Hamiltonian systems, giving definitions of many concepts like the monodromy matrix and the stability indices and providing some basic formulae for these quantities. In Section 3 we define the different stability types that are possible in the general case and we introduce a new terminology for them. In Section 4 we study some particular cases. For completeness sake in Sections 4.1 and 4.2 we review the stability types that appear in Hamiltonian systems with two and three degrees of freedom respectively, while a detailed study of the four degrees of freedom case is done in Section 4.3.

2. A review of the stability theory of Hamiltonian systems with $N+1$ degrees of freedom

Let us consider an autonomous Hamiltonian system H_0 , not necessarily integrable, with $N+1$ degrees of freedom, where N is an integer with $N \geq 1$, which is perturbed. Then its Hamiltonian function can be written as:

$$H = H_0 + \varepsilon H_1 \quad (1),$$

where ε is the perturbation parameter. The equations of motion for this system can be expressed in the form

$$\dot{x} = -J \cdot \nabla H = -J \cdot \nabla (H_0 + \varepsilon H_1) \quad (2),$$

with

$$x = (q_1, q_2, \dots, q_{N+1}, p_1, p_2, \dots, p_{N+1})'$$

and

$$\nabla H = \left(\frac{\partial H}{\partial q_1}, \frac{\partial H}{\partial q_2}, \dots, \frac{\partial H}{\partial q_{N+1}}, \frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial p_2}, \dots, \frac{\partial H}{\partial p_{N+1}} \right)',$$

where q_i , $i=1, 2, \dots, N+1$ are the generalized coordinates and p_i , $i=1, 2, \dots, N+1$ the conjugate momenta and prime (') denotes the transpose matrix. The matrix J has the following block form:

$$J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix} \quad (3),$$

where I_n is the $n \times n$ identity matrix with $n=N+1$ and 0_n is the $n \times n$ matrix with all its elements equal to zero.

The linear stability of a periodic orbit of this system with period T is determined by the solution of the linearized equations:

$$J \cdot \dot{\xi} = (P_0 + \varepsilon P_1) \xi = P \cdot \xi \quad (4),$$

where ξ is a vector denoting the deviation from the given periodic orbit in the $(2N+2)$ -dimensional phase space of the system and represented by a $(2N+2) \times 1$ matrix, $P = P_0 + \varepsilon P_1$ is the Hessian matrix of the Hamiltonian (1) calculated on the periodic orbit the stability of which we study. The elements of matrix P

$$P_{ij} = \frac{\partial^2 H}{\partial x_i \partial x_j}, \quad i, j = 1, 2, \dots, 2N+2 \quad (5)$$

are T -periodic functions of time since the RHS of Eq. (5) is calculated for the T -periodic orbit. Eqs. (4) are the so-called *variational equations* of the system for the particular periodic orbit. A $(2N+2) \times (2N+2)$ matrix whose individual columns consist of $2N+2$ linearly independent solutions of Eqs. (4) is called a *fundamental matrix of solutions* of the variational equations (4). The fundamental matrix $X(t)$ whose solutions correspond to the initial conditions

$$X(0) = I_{2N+2} \quad (6),$$

gives the evolution of the deviation vector ξ for $t = kT$, $k \in \mathbb{N}^*$ through the relation

$$\xi(kT) = [X(T)]^k \cdot \xi(0) \quad (7).$$

The matrix

$$A = X(T) \quad (8)$$

is called the *monodromy matrix* and satisfies the symplectic condition (Yakubovich & Starzhinskii 1975, Hadjidemetriou 1989)

$$A' \cdot J \cdot A = J \quad (9).$$

The stability type of the periodic orbit is determined by knowing the nature of the eigenvalues of the matrix A . Due to the symplectic condition (9) and due to the fact that the matrix coefficients are real, the eigenvalues of matrix A have the following properties: if λ is an eigenvalue then $1/\lambda$ is also an eigenvalue, and if λ is an eigenvalue the complex conjugate λ^* is also an eigenvalue. These properties show that the eigenvalues $\lambda=1$ and $\lambda=-1$ are always double eigenvalues and that complex eigenvalues with modulus not equal to 1 always appear in quartets. When all the eigenvalues are on the unit circle the corresponding periodic orbit is stable. If there exist eigenvalues not on the unit circle the periodic orbit is unstable.

An interesting question is the following: Assume that for $\varepsilon=0$ we know the eigenvalues of the corresponding monodromy matrix A_0 . What are the possibilities for the eigenvalues of the perturbed matrix A , which corresponds to the perturbed system (1) under the symplectic constraint? This question has been answered in the 50's by Krein (1950), Gelfand & Lidskii (1955) and Moser (1958). The theory they developed

is presented in detail by Yakubovich & Starzhinskii (1975). According to this theory the only possible movement of simple eigenvalues on the unit circle, due to perturbation, is movement on the unit circle, which means that the stability type of the periodic orbit does not change. On the other hand multiple eigenvalues on the unit circle have two possibilities. Either they move on the unit circle or off the unit circle. The second case is called complex instability. Finally, if a double eigenvalue equals +1 or -1 then under the effect of perturbation these eigenvalues either remain on the unit circle or move on the real axis off the unit circle. The latter case is called in general instability.

An important quantity in determining the fate of eigenvalues under perturbation is the so-called "kind" of the eigenvalues. Let g be an eigenvector corresponding to the simple eigenvalue λ on the unit circle, then we define the inner product

$$\langle g, g \rangle = i (Jg, g) \quad (10),$$

where $(.,.)$ denotes the usual inner product. Then λ is called an eigenvalue of the first kind if $\langle g, g \rangle > 0$ and of the second kind if $\langle g, g \rangle < 0$. Let now λ be an r -tuple eigenvalue on the unit circle and g a corresponding eigenvector. If $\langle g, g \rangle > 0$ for any eigenvector then λ is called an r -tuple eigenvalue of the first kind and if $\langle g, g \rangle < 0$ then λ is an r -tuple eigenvalue of the second kind. Eigenvalues of the first and second kind are said to be definite. If on the other hand there exists an eigenvector $g \neq 0$ such that $\langle g, g \rangle = 0$ then the eigenvalue is said to be indefinite or of mixed kind. In this case there exist an eigenvector g so that $\langle g, g \rangle$ is positive and another one so that $\langle g, g \rangle$ is negative. The well known theorem of Krein-Gelfand-Lidskii (Yakubovich & Starzhinskii 1975, Hadjidemetriou 1989) states that a linear system is *strongly stable* (i.e. no small perturbation may turn it unstable) if and only if all eigenvalues lie on the unit circle and are definite.

From the above theory we see that if we have only simple eigenvalues on the unit circle then it is impossible to have instability due to small perturbation. The only way to have instability by perturbing the system is to have two simple eigenvalues of different kinds colliding to create a double eigenvalue. Then instability may occur.

A few remarks on the connection of Hamiltonian systems with symplectic maps are necessary. Since autonomous Hamiltonian systems are conservative, the constancy of the Hamiltonian function (1) introduces a constraint of the form:

$$H(q_1, q_2, \dots, q_{N+1}, p_1, p_2, \dots, p_{N+1}) = c \quad (11),$$

where c is a constant value. This constraint fixes an eigenvalue to be equal to 1 and so by the symplectic nature of the problem there must be a second eigenvalue equal to 1. Thus there are only $2N$ non-constant eigenvalues. So we can constrain the study of a $N+1$ degrees of freedom Hamiltonian systems to a $2N$ -dimensional subspace of the general phase space. This subspace is obtained by the well-known method of the *Poincaré surface of section (PSS)* (see for example Lichtenberg & Lieberman 1983). Generally speaking we can assume a PSS of the form $q_{N+1} = \text{constant}$. Then only the variables $q_1, q_2, \dots, q_N, p_1, p_2, \dots, p_N$ are needed to describe the evolution of an orbit on the PSS, since p_{N+1} can be found by solving Eq. (11). The corresponding monodromy matrix of the periodic orbit is also symplectic and will be denoted as L . In this sense a $N+1$ degrees of freedom Hamiltonian system corresponds to $2N$ -dimensional symplectic map.

The eigenvalues of L define the stability of the corresponding periodic orbit. These eigenvalues are roots of the *characteristic polynomial*:

$$P(\lambda) = \det (L - \lambda I_{2N}) \quad (12),$$

which is a palindrome of the form (Howard & MacKay 1987)

$$P(\lambda) = \lambda^{2N} - A_{N-1} \lambda^{2N-1} + A_{N-2} \lambda^{2N-2} + \dots + (-1)^N A_0 \lambda^N + \dots - A_{N-1} \lambda + 1 \quad (13).$$

The coefficients of P can be easily expressed as functions of the elements of matrix L . The characteristic polynomial (13) can be written in a simpler form in terms of the *stability index*:

$$b = \frac{1}{\lambda} + \lambda \quad (14).$$

In particular it becomes

$$Q(b) = A'_0 b^N - A'_1 b^{N-1} + \dots + (-1)^{N-1} A'_{N-1} b + (-1)^N A'_N \quad (15).$$

The polynomial $Q(b)$ is called the *reduced characteristic polynomial*. One of the main advantages of introducing the stability indices b_i , $i=1,2,\dots,N$ is that they solve a polynomial of half the original order, i.e. a polynomial equation of order N . This turns the computational problem into a much more tractable one.

The coefficients A'_i , $i=0,1,2,\dots,N$ of $Q(b)$ are related to the roots b_i , $i=1,2,\dots,N$ by the formulae:

$$\begin{aligned} A'_0 &= 1 \\ A'_1 &= \sum_{i=1}^N b_i \\ A'_2 &= \sum_{i<j} b_i b_j \\ &\vdots \\ A'_N &= b_1 b_2 \dots b_N \end{aligned} \quad (16).$$

So A'_i is the sum of all possible i -tuples of b_1, \dots, b_n . The connection between the coefficients A_i , $i=0,1,\dots,N-1$ of the characteristic polynomial (13) and the coefficients A'_i , $i=0,1,2,\dots,N$ of the reduced characteristic polynomial (15) can be found using some algebra and induction. In particular we get:

$$A_{N-i} = \sum_{u=0}^{[i/2]} \binom{N-i+2u}{u} A'_{i-2u}, \quad i = 1, 2, \dots, N \quad (17),$$

where $[i]$ denotes the integer part of i and $\binom{i}{j}$ denotes the combinations of i over j . The stability type of a periodic orbit is represented by a point in the N -dimensional parameter space \mathcal{S} whose coordinates are the coefficients A_0, A_1, \dots, A_{N-1} of the characteristic polynomial $P(\lambda)$.

3. Stability types of periodic orbits

The configuration of the eigenvalues of the monodromy matrix L on the complex plane, or equivalently the values of the stability indices, determine the stability type of a periodic orbit. In particular we have the cases:

- The orbit is stable (S) when $b \in (-2, 2)$, which means that λ and $1/\lambda$ are complex conjugate numbers on the unit circle.
- The orbit is unstable (U) when $b \in (-\infty, -2) \cup (2, \infty)$, which means that λ and $1/\lambda$ are real. We remark that the cases $b > 2$ and $b < -2$ are equivalent regarding the stability character of the periodic orbit, but not completely identical since a positive b cannot become negative under a continuous change of a parameter of the system.
- The orbit is complex unstable (Δ) when $b \in \mathbb{C} - \mathbb{R}$, which means that we have four complex eigenvalues not laying on the unit circle and the real axis, forming two pairs of inverse numbers and two pairs of complex conjugate numbers. Two of the eigenvalues are inside the unit circle while the other two are outside it.

All these different cases are shown in Fig. 1.

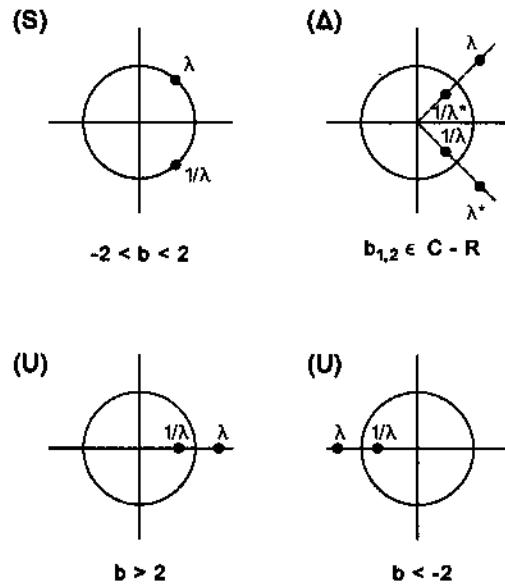


Figure 1 Configuration of the eigenvalues of the monodromy matrix on the complex plane, with respect to the unit circle, for the stable (S), unstable (U) and complex unstable (Δ) cases. In every case b is the corresponding stability index. We remark that λ^* denotes the complex conjugate of λ .

The general stability type of a periodic orbit of a Hamiltonian system with $N+1$ degrees of freedom, or of a $2N$ -dimensional symplectic map is

$$S_n U_m \Delta_l \quad (18)$$

with n , m and l integer numbers, denoting that $2n$ eigenvalues are on the unit circle, $2m$ eigenvalues are on the real axis and $4l$ eigenvalues are on the complex plane but not on the unit circle and the real axis. In order to distinguish between the different arrangements of the eigenvalues on the real axis we can use the notation

$$S_n U_{m_1, m_2} \Delta_l \quad (19)$$

with $m=m_1+m_2$, denoting the case of having $2m_1$ negative real eigenvalues and $2m_2$ positive real eigenvalues. The integers n, m, l satisfy the inequalities:

$$0 \leq n \leq N, \quad 0 \leq m \leq N, \quad 0 \leq l \leq [N/2] \quad (20),$$

and the constraint

$$n + m + 2l = N \quad (21).$$

We note that a periodic orbit is stable only when its stability type is S_N . In all other cases the orbit is unstable, since there exist eigenvalues not on the unit circle.

4. Cases of definite number of degrees of freedom

In this section we study some particular cases, namely Hamiltonian systems with two, three and four degrees of freedom. Although the stability properties of Hamiltonian systems with two and three degrees of freedom are well known (e.g. Hénon 1965, Broucke 1969) we include a brief treatment of these cases for completeness sake and in order to illustrate the use of the terminology of the different stability types introduced the previous section. On the other hand the study of systems with four degrees of freedom is far from considered complete. So we do a detailed study of all the possible stability types and the transitions between them.

4.1. The case of two degrees of freedom

In the case of a Hamiltonian system with two degrees of freedom we have $N+1=2$ so the characteristic polynomial (13) becomes

$$P(\lambda) = \lambda^2 - A_0 \lambda + 1 \quad (22),$$

where the coefficient A_0 is the trace of the monodromy matrix L

$$A_0 = \text{Tr}L \quad (23).$$

So, there exists only one stability index b_1 , hence the reduced characteristic polynomial (15) is simply

$$Q(b) = b - A_1' \quad (24),$$

where

$$b_1 = A_1' = A_0 \quad (25),$$

as we derive from Eqs. (16) and (17), for $N=1$. So the stability type of the orbit depends on the value of A_0 . In particular we have only 2 types of stability. If $|b_1|=|A_0|<2$ the orbit is stable (S_1) and if $|A_0|>2$ unstable (U_1). The usual notation for these two stability types is S for stable and U for unstable (e.g. Greene 1968, Contopoulos 1970a;b, Bountis 1981), which does not practically differ from our notation.

The parameter space \mathcal{S} , where the possible stability types are defined is one-dimensional since the only coordinate is A_0 (Fig. 2). In this space we have 3 different stability regions. The transition boundaries between different stability regions correspond to the constraints $b_1=-2$ and $b_1=2$, so that the possible regions are: a) $A_0 < -2$, which corresponds to the $U_{1,0}$ stability type, b) $-2 < A_0 < 2$, which corresponds to the S_1 stability type, and c) $A_0 > 2$ which corresponds to the $U_{0,1}$ stability type. The only possible transition is $S \rightarrow U$ ($U_{1,0}$ or $U_{0,1}$) and vice versa. These transitions are performed when the point A , which represents the stability type of the periodic orbit, passes through the points $A_0 = \pm 2$. The transition $U_{1,0} \rightarrow U_{0,1}$ is not possible since the two regions in \mathcal{S} are not neighboring.

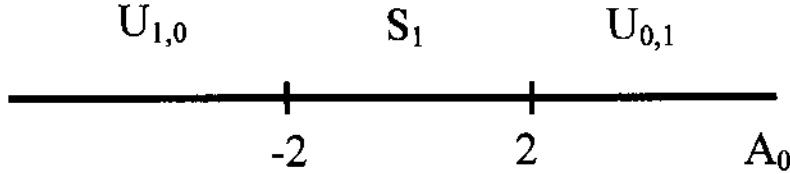


Figure 2 The one-dimensional parameter space \mathcal{S} of a Hamiltonian system with two degrees of freedom. A_0 is the coefficient of the corresponding characteristic polynomial. There exist three different stability types $U_{1,0}$, S_1 , $U_{0,1}$. The boundaries between the different stability regions are the points $A_0 = 2$ and -2 , which correspond to the stability index b being equal to 2 and -2 , respectively.

4.2. The case of three degrees of freedom

In this case $N+1=3$ so the characteristic polynomial (13) is

$$P(\lambda) = \lambda^4 - A_1 \lambda^3 + A_0 \lambda^2 - A_1 \lambda + 1 \quad (26).$$

The coefficient A_0 and A_1 are related to the elements of the monodromy matrix L through the relations

$$A_0 = \sum_{i < j} \begin{vmatrix} L_{ii} & L_{ij} \\ L_{ji} & L_{jj} \end{vmatrix}, \quad A_1 = \text{Tr}L \quad (27).$$

The two stability indices b_1, b_2 are roots of the reduced characteristic polynomial (15)

$$Q(b) = b^2 - A'_1 b + A'_2 = 0 \quad (28).$$

The coefficients of the two polynomials are related through

$$A'_1 = A_1 = b_1 + b_2, \quad A'_2 = A_0 - 2 = b_1 b_2 \quad (29)$$

as we derive from Eqs. (16) and (17) for $N=2$. So the two stability indices are

$$b_1, b_2 = \frac{A_1 \pm \sqrt{A_1^2 - 4(A_0 - 2)}}{2} \quad (30).$$

The parameter space \mathcal{S} where the possible stability types are defined, is two-dimensional with coordinates the coefficients A_0 and A_1 . The transition boundaries between the different stability regions in \mathcal{S} are given by substituting $b=\pm 2$ in Eq. (28) and using Eqs. (29), which yields the lines:

$$b = +2 \Rightarrow A_0 = 2A_1 - 2, \quad b = -2 \Rightarrow A_0 = -2A_1 - 2 \quad (31)$$

and by forcing $b_1=b_2$ in (29) or equivalently putting the discriminant of (28) equal to zero, which yields the curve

$$A_0 = \frac{1}{4} A_1^2 + 2 \quad (32).$$

These boundaries create the 7 regions of different stability types seen in Fig. 3.

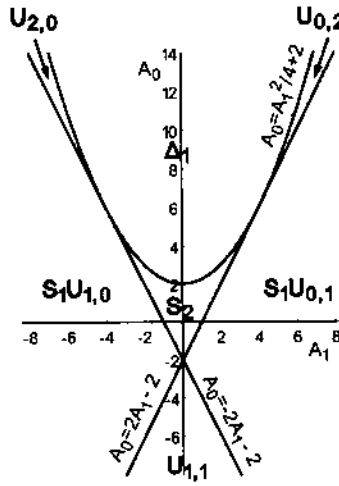


Figure 3 The parameter space \mathcal{S} of a Hamiltonian system with three degrees of freedom. A_0 and A_1 are the coefficients of the characteristic polynomial. On every stability region the corresponding stability type is marked.

We note that Broucke (1969) performed the above analysis in his work on the elliptic restricted three-body problem, where he introduced a notation different to ours for the possible stability types. We underline the fact that Broucke gave different names for all the regions seen in Fig. 3 taking into account the different configuration of the eigenvalues on the real axis. In a similar approach Dullin & Meiss (1998) gave different names to the 7 stability types of a four-dimensional symplectic map. On the other hand, most authors do not take into account the different arrangements of the eigenvalues on the real axis, referring only to 4 different cases (e.g. Contopoulos 1983, Pfenniger 1984, Contopoulos & Magnenat 1985, Patsis & Zachilas 1994, Vrahatis et al. 1996, 1997). Although our terminology may seem a little heavy for the cases of two and three degrees of freedom, in comparison to the already used terminology, it is perfectly suited for systems with many degrees of freedom, since it gives in a very clear way all the information needed for the configuration of the eigenvalues on the complex plane.

4.3. The case of four degrees of freedom

In this case $N+1=4$ so the characteristic polynomial (13) is

$$P(\lambda) = \lambda^6 - A_2 \lambda^5 + A_1 \lambda^4 - A_0 \lambda^3 + A_1 \lambda^2 - A_2 \lambda + 1 \quad (33),$$

where the coefficient A_0 , A_1 and A_2 are related to the elements of the monodromy matrix L through the relations

$$A_0 = \sum_{i<j<k} \begin{vmatrix} L_{ii} & L_{ij} & L_{ik} \\ L_{ji} & L_{jj} & L_{jk} \\ L_{ki} & L_{kj} & L_{kk} \end{vmatrix}, \quad A_1 = \sum_{i<j} \begin{vmatrix} L_{ii} & L_{ij} \\ L_{ji} & L_{jj} \end{vmatrix}, \quad A_2 = \text{Tr}L \quad (34).$$

The three stability indices b_1 , b_2 , b_3 are roots of the reduced characteristic polynomial (15):

$$Q(b) = b^3 - A_1' b^2 + A_2' b - A_3' = 0 \quad (35).$$

For simplicity we use $A \equiv A_2$, $B \equiv A_1$, $C \equiv A_0$. Then Eqs. (16) and (17) give

$$\begin{aligned} A &= A_1' = b_1 + b_2 + b_3 \\ B &= A_2' + 3 = b_1 b_2 + b_1 b_3 + b_2 b_3 + 3 \\ C &= A_3' + 2A = b_1 b_2 b_3 + 2(b_1 + b_2 + b_3) \end{aligned} \quad (36)$$

The discriminant Δ of $Q(b)$ is

$$\Delta = q^2 + 4p^3 \quad (37)$$

where

$$q = -2A^3/27 + A(B-3)/3 - C + 2A, \quad p = (B-3)/3 - A^2/9 \quad (38).$$

So, the three stability indices b_1 , b_2 , b_3 are

$$\begin{aligned} b_1 &= z_1 + z_2 + \frac{A}{3} \\ b_2 &= -\frac{z_1 + z_2}{2} + \frac{A}{3} + i \frac{\sqrt{3}}{2} (z_1 - z_2) \\ b_3 &= -\frac{z_1 + z_2}{2} + \frac{A}{3} - i \frac{\sqrt{3}}{2} (z_1 - z_2) \end{aligned} \quad (39)$$

with

$$z_1 = \sqrt[3]{\frac{-q + \sqrt{\Delta}}{2}}, \quad z_2 = \sqrt[3]{\frac{-q - \sqrt{\Delta}}{2}} \quad (40).$$

The parameter space \mathcal{S} where the possible stability types are defined is the three-dimensional space (A,B,C). The regions of the possible stability types in \mathcal{S} , are defined by the transition boundaries that correspond to the following constraints:

- One stability index is equal to 2. The corresponding boundary is plane p_1 shown in Fig. 4(a). The equation of this plane is obtained by putting $b=+2$ in Eq. (35) and using Eqs. (36):

$$C = 2(1 - A + B) \quad (41).$$

- One stability index is equal to -2. The boundary surface is the plane p_2 shown in Fig. 4(b). This plane is defined by the equation

$$C = -2(1 + A + B) \quad (42).$$

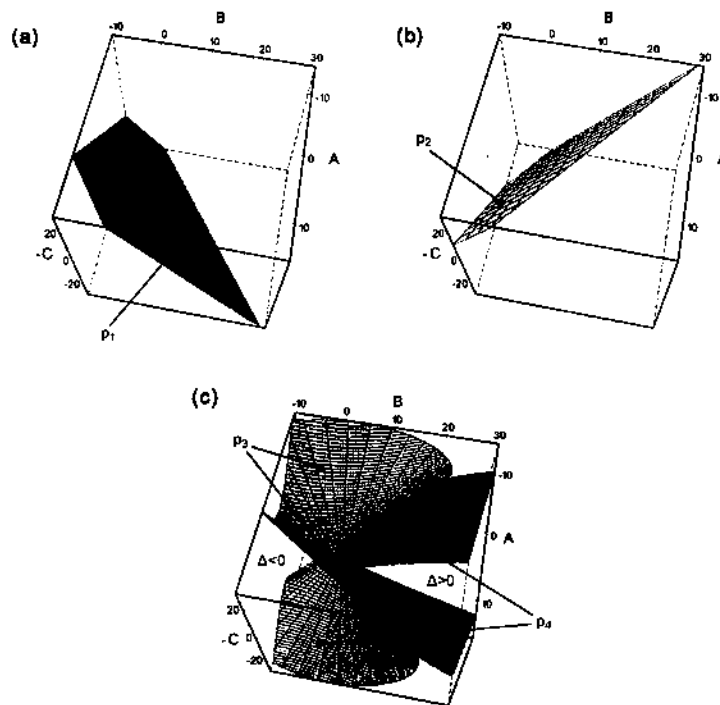


Figure 4 The boundaries of all the possible stability regions in the parameter space \mathcal{S} of a Hamiltonian system with four degrees of freedom. A, B, C are the coefficients of the corresponding characteristic polynomial. The constraint $b=+2$ corresponds to the plane p_1 (a), while the constraint $b=-2$ to the plane p_2 (b). The equality of at least two stability indices corresponds to the two-sheeted surface p_Δ in (c) composed of the p_3 and p_4 surfaces. In (c) the regions where the discriminant Δ of the reduced characteristic polynomial is positive and negative are marked.

- At least two stability indices are equal to each other. This condition is equivalent to the discriminant Δ (Eq. 37) being equal to zero. The corresponding boundary surface p_Δ is the two-sheeted surface shown in Fig. 4(c). The upper sheet in Fig. 4(c) is denoted as p_3 and the lower sheet as p_4 . The equations of the two sheets are obtained by putting $\Delta=0$ in Eq. (37) and using Eqs. (38). So we get:

$$C = -\frac{2A^3}{27} + \frac{A(B-3)}{3} + 2A \pm 2 \sqrt{-\left[\frac{B-3}{3} - \frac{A^2}{9}\right]^3} \quad (43),$$

with '+' corresponding to p_3 and '-' to p_4 . This surface divides the parameter space \mathcal{S} in the two regions shown in Fig. 4(c). In the region seen in Fig. 4(c) on the left side of p_Δ , the stability indices are three distinct real numbers ($\Delta < 0$), while in the region seen on the right side of p_Δ , we have one real stability index with the other two indices being complex conjugate numbers ($\Delta > 0$). On the surface p_Δ ($\Delta = 0$) the stability indices are real with at least two of them equal to each other. Along the curve where the two sheets p_3 and p_4 join the three stability indices are real and equal.

The transition boundaries p_1 , p_2 and p_Δ create 13 regions of different stability types. Some of these regions are shown in Fig. 5. Fig. 5(a) is produced by the superposition of the three frames seen in Fig. 4. In Fig. 5(a) the regions of 8 stability types are shown, in particular the regions of the stability types: $S_1U_{2,0}$, $S_2U_{1,0}$, $U_{1,0}\Delta_1$, $U_{1,2}$, $S_1U_{1,1}$, $S_2U_{0,1}$, $S_1U_{0,2}$ and $S_1\Delta_1$. In Fig. 5(b) the same portion of the parameter space \mathcal{S} to the one shown in Fig. 5(a), is seen from a different point of view so that the regions of the stability types $U_{2,1}$ and $U_{0,1}\Delta_1$ are also visible. We note that new types of instabilities are introduced in Hamiltonian systems with four degrees of freedom and in particular the types $S_1\Delta_1$, $U_{1,0}\Delta_1$ and $U_{0,1}\Delta_1$ where we have the coexistence of complex instability with stable and unstable configuration.

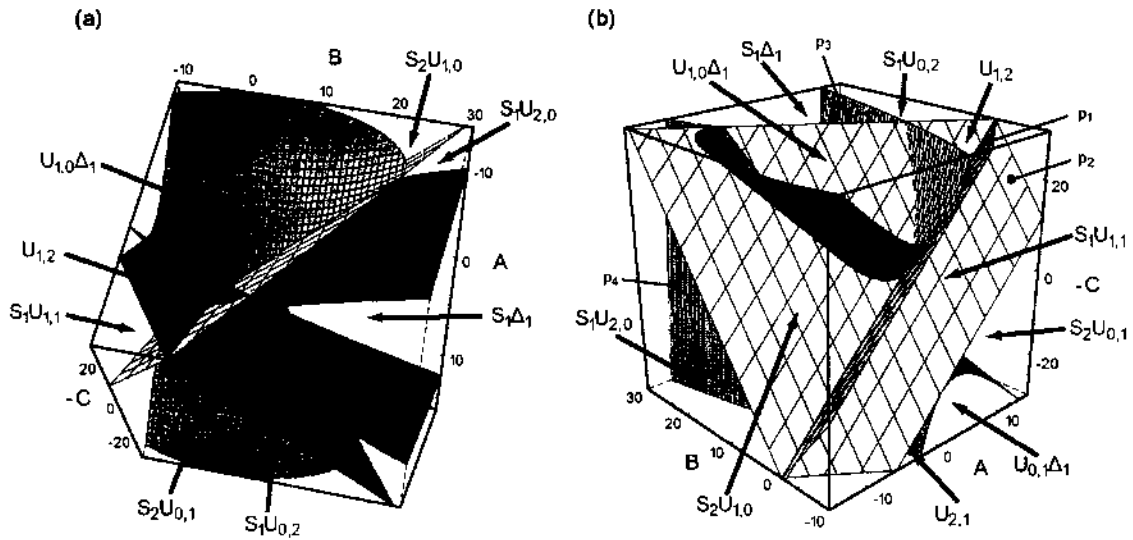


Figure 5 The parameter space \mathcal{S} of a Hamiltonian system with four degrees of freedom. A , B and A are the coefficients of the characteristic polynomial. On every stability region the corresponding stability type is marked. (a) The regions of the stability types: $S_1U_{2,0}$, $S_2U_{1,0}$, $U_{1,0}\Delta_1$, $U_{1,2}$, $S_1U_{1,1}$, $S_2U_{0,1}$, $S_1U_{0,2}$ and $S_1\Delta_1$. (b) The same portion of the parameter space \mathcal{S} to the one shown in frame (a), seen from a different point of view so that the regions of the stability types $U_{2,1}$ and $U_{0,1}\Delta_1$ are also visible. The planes p_1 , p_2 and the surfaces p_3 , p_4 are also marked.

We note that certain transitions between different stability types are not possible. For example the transition $S_1U_{2,0} \rightarrow S_1U_{0,2}$ cannot happen, since the corresponding regions in \mathcal{S} are not neighboring as seen in Figs. 5(a) and 5(b). Some transitions are possible when very specific conditions are satisfied, since they correspond to the crossing of a particular point in \mathcal{S} , like the transition $S_3 \rightarrow U_{3,0}$. On the other hand other transitions are performed by the crossing of certain curves, like the transition $S_2U_{0,1} \rightarrow U_{1,2}$ which is performed by the crossing of a line as seen in Fig. 5(a), or by the crossing of a surface in \mathcal{S} , like for example the transition $S_2U_{0,1} \rightarrow S_1U_{0,2}$ since the two regions are separated by plane p_1 as seen in Fig. 5(a). We remark that the region that corresponds to the S_3 type is directly connected to all other regions. Thus, the direct transition from the stable case S_3 to any unstable type is possible. A more detailed study of the configuration of the parameter space \mathcal{S} can be found in Skokos (2001).

Acknowledgements

This work was supported by the European Union in the framework of EPIET II and KΠΣ 1994-1999, by the Research Committee of the Academy of Athens and by the Association EURATOM-Hellenic Republic under the contract ERB 5005 CT 99 0100.

References

1. Bountis T. C., 1981, *Physica D*, 3, 577
2. Broucke R. A., 1969, NASA Technical Report 32, 1360
3. Contopoulos G., 1970a, *Astron. J.*, 75, 96
4. Contopoulos G., 1970b, *Astron. J.*, 75, 108
5. Contopoulos G., 1983, *Lett. Nuovo Cimento*, 38, 257
6. Contopoulos G. & Magnenat P., 1985, *Celest. Mech.*, 37, 387
7. Dullin H. R. & Meiss J. D., 1998, *Phys. Lett. A*, 247, 227
8. Gautero J. L. & Froeschlé C., 1990, *Nuovo Cimento*, 105, 657
9. Gelfand I. M. & Lidskii V. B., 1955, *Ups. Matem. Nauk*, 10, 3
10. Greene J. M., 1968, *J. Math. Phys.*, 9, 760
11. Hadjidemetriou J., 1975, *Celest. Mech.*, 12, 255
12. Hadjidemetriou J., 1989, in 'Order and chaos in non-linear dynamical systems', Vol. II, eds. Bountis T. C. & Pnevmatikos St., Pnevmatikos, Athens, 1 (in Greek).
13. Heggie D. C., 1985, *Celest. Mech.*, 35, 357
14. Hénon M., 1965, *Annal Astrphysique*, 28, 992
15. Howard J. E. & Dullin H. R., 1998, *Phys. Lett. A*, 246, 273
16. Howard J. E. & MacKay R. S., 1987, *J. Math. Phys.*, 28, 1036
17. Krein M. G., 1950, *Dokl. Akad. Nauk SSSR*, 73, 445
18. Lichtenberg A. J. & Leiberman M. A., 1983, *Regular and stochastic motion*, Springer-Verlag
19. Moser J., 1958, *Commun. Pure Appl. Math.*, XI, 81
20. Patsis P. A. & Zachilas L., 1994, *Int. J. Bif. Chaos*, 4, 1399
21. Pfenniger D., 1984, *Astron. Astroph.*, 134, 373
22. Skokos Ch., 2001, *Physica D*, 159, 155
23. Yakubovich V. A. & Starzhinskii V. M., 1975, *Linear Differential Equations with Periodic Coefficients*, Vol. 1, J. Wisley, New York
24. Vrahatis M. N., Bountis T. C. & Kollmann M., 1996, *Int. J. Bif. Chaos*, 6, 1425
25. Vrahatis M. N., Isliker H. & Bountis T. C., 1997, *Int. J. Bif. Chaos*, 7, 2707